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Some finite p-groups with bounded index of every cyclic subgroup in its normal closure ☆

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ABSTRACT

Let G be a finite p-group. G is called a $BI(p^m)$-group if $|\langle a\rangle^G : \langle a\rangle| \leqslant p^m$ for every $a \in G$. In this paper, we mainly study the properties of $BI(2)$-groups and $BI(p^2)$-groups for $p \geqslant 3$, and we study some $BI(p^m)$-groups ($m \geqslant 3$) with some additional requirement.

1. Introduction

A group G is called a $BI(l)$-group ($G \in BI(l)$ for short) if $|\langle a\rangle^G : \langle a\rangle| \leqslant l$ for every $a \in G$. As only finite p-groups are considered in this paper, we just investigate $BI(p^m)$-groups. Note that every p-group is a $BI(p^m)$-group for some integer m. It is difficult to study the $BI(p^m)$-groups if there is not restriction on m. So we will study $BI(p)$-groups, $BI(p^2)$-groups and $BI(p^m)$-groups ($m \geqslant 3$) with some additional requirement.

Recall that a group G is called Dedekind group if every subgroup of G is normal in G. There are some generalizations for Dedekind groups (for instance, see [4,5]). It is clear that $G \in BI(1)$ if and only if G is a Dedekind group. Hence the $BI(l)$-groups can also be seen as a kind of generalization of Dedekind groups.

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It is clear that $BI(l)$-property is inherited by subgroups and quotient groups. But by Theorem 4.7, the direct product of two $BI(p^m)$ may be a non-$BI(p^m)$-group. Hence $BI(l)$-property is not closed with respect to extension of its member.

M. Herzog, P. Longobardi, M. Maj and A. Mann [5] have investigated the so-called J-group G ($G \in J$ for short), in which each element $x \in G$ satisfies either $\langle x\rangle \lhd G$ or $\langle x, x^g\rangle \lhd G$ for all $g \in G - N_G(\langle x\rangle)$. If we consider only finite p-groups, it is clear that a $BI(p)$-group must be a J-group. But there exists a 2-group $G \in J$ which is not a $BI(2)$-group (Example 2.1). However, by Proposition 2.2, J-property and $BI(p)$-property are the same thing for $p \geqslant 3$.

In Section 2, we investigate $BI(p)$-groups. By Proposition 2.2 and Example 2.1, we can concentrate on $BI(2)$-groups. Using the lemmas on $BI(p)$-groups, we conclude that $exp(G) = 4$ or 8 if $G \in BI(2)$ and $cl(G) = 3$ (Theorem 2.8). This result is not included in [5], although the other case that $G \in J$ and $cl(G) = 3$ with $p \geqslant 3$ was studied [5].

In Section 3, we study the group $G \in BI(p^2)$ for $p \geqslant 3$. We prove that $G^{p^2} \leqslant Z(G)$, $exp(G') \leqslant p^2$ and $cl(G) \leqslant 4$ (Theorems 3.4 and 3.5). And we find that G^{p^2} is cyclic if $cl(G) = 4$ (Theorem 3.8). In Section 4, we study some special $BI(p^m)$-groups. Theorem 4.1 gives us some information of regular $BI(p^m)$-groups with $p \geqslant 3$. We also consider the case that $G \in BI(p^m)$ and $exp(G) = p$, $cl(G) = m + 1$ for $p \geqslant 3$. In this section, a very interesting $BI(p^m)$-group is given in Example 4.4.

We use standard notation. Throughout this paper, G denotes a finite p-group for some prime p. $H \lhd G$ means that H is a normal subgroup of G. $\langle a\rangle^G$ is the normal closure of $\langle a\rangle$. $\Omega_n(G) = \langle x \in G \mid x^{p^n} = 1\rangle$, $G^{p^n} = \langle x^{p^n} \mid x \in G\rangle$. $cl(G)$ is the nilpotent class of G. $exp(G)$ is the exponent of G. $\gamma_i(G), Z_i(G)$ is the lower central series, the upper central series of G, respectively. $Z(G) = Z_1(G)$. $G' = \gamma_2(G)$. $N \rtimes H$ means the semidirect product of N and H, where $N \lhd N \rtimes H$.

2. *BI*(2)-groups

It is clear that a finite p-group $G \in BI(p)$ implies $G \in J$. But there is a 2-group $G \in J$ which is not a $BI(2)$-group.

Example 2.1. Let $G = \langle a, b \mid a^4 = 1,\ b^{2^4} = 1,\ a^b = a^3 b^{2^2},\ [a, b^2] = 1\rangle$. Then $|G| = 64$, $\langle a\rangle \cap \langle b\rangle = 1$. It is easy to check that $G \in J$ (this can also be done by GAP [3]). But $|\langle a\rangle^G : \langle a\rangle| = 4$, and then $G \notin BI(2)$.

However, if $p \geqslant 3$, we can prove that $G \in J$ if and only if $G \in BI(p)$.

Proposition 2.2. *Suppose $p \geqslant 3$. Then $G \in J$ if and only if $G \in BI(p)$.*

Proof. We need only to prove that $G \in J$ implies $G \in BI(p)$. Assume $G \in G$ and $x \in G$ such that $\langle x\rangle$ is not normal in G. Then [5, Lemma 7] implies that $\langle x\rangle$ is normal in $\langle x\rangle^G = \langle x, x^g\rangle = \langle x, [x, g]\rangle$, where $g \in G - N_G(\langle x\rangle)$. By [5, Theorem 13], $[x, g]$ is of order p and so $|\langle x\rangle^G : \langle x\rangle| \leqslant p$. □

By Proposition 2.2, only the case $p = 2$ is need be considered for $BI(p)$-group. But since the proofs of the following three lemmas have no difference between $p = 2$ and $p \geqslant 3$, we do not drop the case $p \geqslant 3$ from these lemmas.

Lemma 2.3. *Let $G \in BI(p)$ with $exp(G) = p$. Then $cl(G) \leqslant 2$.*

Proof. Since G is a $BI(p)$-group and $exp(G) = p$, $|\langle a\rangle^G| \leqslant p^2$ for every $a \in G$. Thus $\langle a\rangle^G \leqslant Z_2(G)$, and $a \in Z_2(G)$, which implies $cl(G) \leqslant 2$. □

Lemma 2.4. *Let $G = \langle a, b\rangle$ be noncyclic. If $|\langle a\rangle^G : \langle a\rangle| \leqslant p$, $|\langle b\rangle^G : \langle b\rangle| \leqslant p$ and $\langle a\rangle \cap \langle b\rangle = 1$, then $cl(G) \leqslant 2$ and $|G'| \leqslant p$.*

Proof. Since $G' \leqslant \langle a\rangle^G \cap \langle b\rangle^G$ and $|\langle a\rangle^G \cap \langle b\rangle^G| \leqslant p^2$, we need only to consider the case that $|\langle a\rangle^G \cap \langle b\rangle^G| = p^2$. Hence $|\langle a\rangle \cap \langle b\rangle^G| = p$ and $|\langle b\rangle \cap \langle a\rangle^G| = p$.

If $|a| = p$, then $\langle a\rangle \cap \langle b\rangle^G = \langle a\rangle$, and $\langle a\rangle \leqslant \langle b\rangle^G$. Thus $G = \langle b\rangle^G\langle a\rangle = \langle b\rangle^G$. But $\langle b\rangle^G \leqslant \langle b, G'\rangle$, and $G = \langle b\rangle$, which contradicts to the fact that G is noncyclic. Hence $|a|, |b| > p$. Let $\langle a_1\rangle = \langle a\rangle \cap \langle b\rangle^G$ and $\langle b_1\rangle = \langle b\rangle \cap \langle a\rangle^G$. Then $\langle a_1\rangle$, $\langle b_1\rangle$ are the unique subgroups of order p in $\langle a\rangle$, $\langle b\rangle$, respectively. For any $x \in G$, we see that $|\langle a\rangle^G : \langle a\rangle \cap \langle a\rangle^x| \leqslant p^2$, and $|\langle a\rangle : \langle a\rangle \cap \langle a\rangle^x| \leqslant p$. Hence $\langle a\rangle \cap \langle a\rangle^x$ is subgroup of a cyclic group $\langle a\rangle$ of order $\geqslant p$. Thus $\langle a_1\rangle \leqslant \langle a\rangle \cap \langle a\rangle^x \leqslant \langle a\rangle^x$. This means $\langle a_1\rangle \lhd G$ and $\langle a_1\rangle \leqslant Z(G)$. Similarly, $\langle b_1\rangle \leqslant Z(G)$.

Since $\langle a\rangle \cap \langle b\rangle = 1$, we have $\langle a_1\rangle \cap \langle b_1\rangle = 1$ and $\langle a\rangle^G \cap \langle b\rangle^G = \langle a_1, b_1\rangle$ is a subgroups of $Z(G)$ and is an elementary abelian subgroup of order p^2. Then $G' \leqslant \langle a\rangle^G \cap \langle b\rangle^G \leqslant Z(G)$, and $cl(G) \leqslant 2$. Thus $G' = \langle [a, b]\rangle$ is of order p. □

Hence, by the above lemma, if $G = \langle a, b\rangle$ is a $BI(p)$-group with $\langle a\rangle \cap \langle b\rangle = 1$, then $cl(G) \leqslant 2$ and $|G'| \leqslant p$.

By [5, Proposition 9], if $G \in J$, then $|G : N_G(\langle a\rangle)| \leqslant p$. For $BI(p^k)$-group, we have the following:

Lemma 2.5. *Let $G \in BI(p^k)$. Then for all $a \in G$, $|G : N_G(\langle a\rangle)| \leqslant p^k$.*

Proof. Let $|a| = p^n$. Since G is a $BI(p^k)$ p-group, $|\langle a\rangle^G| \leqslant p^{n+k}$. Hence there are at most $(p^{n+k} - p^{n-1})/(p^n - p^{n-1}) = p^k + \cdots + 1$ cyclic subgroups of order p^n in $\langle a\rangle^G$. Since $\langle a\rangle^g \leqslant \langle a\rangle^G$ for every $g \in G$, we see $|G : N_G(\langle a\rangle)| \leqslant p^k$. □

From now on in this section, we focus on a $BI(2)$-group G. By [5, Lemma 8], $\Omega_1(G) \leqslant Z_2(G)$ for G is obviously a J-group. Hence, if $|a| = 2^n > 2$, we see that $a^{2^{n-1}} \in Z_2(G)$. But the following lemma shows that $a^{2^{n-1}} \in Z(G)$.

Lemma 2.6. *Let $G \in BI(2)$. Then $a^{2^{n-1}} \in Z(G)$ for any $a \in G$ with $|a| = 2^n > 2$.*

Proof. Suppose $b \in G$ such that $ab \neq ba$. Let $H = \langle a, b\rangle$. If $\langle a\rangle \cap \langle b\rangle \neq 1$, then $a^{2^{n-1}} \in \langle a\rangle \cap \langle b\rangle$. Thus $[a^{2^{n-1}}, b] = 1$. If $\langle a\rangle \cap \langle b\rangle = 1$, by Lemma 2.4, we have $cl(H) = 2$ and $|H'| = 2$. We also have $[a^{2^{n-1}}, b] = [a, b]^{2^{n-1}} = 1$. Thus $a^{2^{n-1}} \in Z(G)$. □

Lemma 2.7. *Let $G = \langle a, b\rangle$ be a $BI(2)$-group with $|a| = 2^n \geqslant 2^4$ and $|b| = 2^m \leqslant 2^n$. If b is an element of minimal order such that $G = \langle a, b\rangle$, then $\langle a\rangle \cap \langle b\rangle = 1$, $cl(G) \leqslant 2$ and $|G'| \leqslant 2$.*

Proof. By Lemma 2.4, it suffices to prove that $\langle a\rangle \cap \langle b\rangle = 1$. Suppose that $\langle a\rangle \cap \langle b\rangle \neq 1$. Then $a^{2^{n-1}} = b^{2^{m-1}}$. We consider the following three cases.

(i) $m \leqslant n - 2$. By [5, Corollary 15], $a^4 \in Z(G)$. Then $b_1 = a^{2^{n-m}}b$ is of order $\leqslant 2^{m-1}$. Obviously $G = \langle a, b_1\rangle$, which contradicts to the choice of b.

(ii) $m = n$. By [5, Proposition 10], $cl(G) \leqslant 3$. By Hall–Petrescu formula,

$$(ab)^{2^{n-1}} = a^{2^{n-1}} b^{2^{n-1}} c_2^{2^{n-1}(2^{n-1}-1)/2} c_3^{2^{n-1}(2^{n-1}-1)(2^{n-1}-2)/6},$$

where $c_2 \in \gamma_2(G)$, $c_3 \in \gamma_3(G)$. By [5, Proposition 12], $exp(G') \leqslant 4$. Hence $(ab)^{2^{n-1}} = 1$. This implies $|ab| < |b|$. Clearly, $G = \langle a, b\rangle = \langle a, ab\rangle$, which contradicts to the choice of b.

(iii) $m = n - 1$. If $n \geqslant 5$, we see $|a^2b| < |b|$ and $G = \langle a, b\rangle = \langle a, a^2b\rangle$. This contradicts with the choice of b. Hence we need only to consider that $n = 4$, $m = 3$. By Lemma 2.5, $a^2 \in N_G(\langle b\rangle)$. By [5, Corollary 15], $b^{a^4} = b$. Hence a^2 induces an automorphism of order $\leqslant 2$ on $\langle b\rangle$. By [5, Proposition 10], $\langle a^2, b^2\rangle$ is abelian. Therefore $b^{a^2} = b^{1+4k}$ for some k. We have that $(ba^2)^2 = b^{2+4k}a^4$, and $(ba^2)^4 = b^4a^8 = 1$. As $G = \langle a, b\rangle = \langle a, ba^2\rangle$, we also get a contradiction. □

Let $G \in BI(p)$. Hence G is a J-group, and then $cl(G) \leqslant 3$ [5, Proposition 10]. Now we consider that case that $cl(G) = 3$. If $p \geqslant 3$ and $cl(G) = 3$, by [5, Proposition 21], then $p = 3$ and $exp(G) = 3^2$. For the case that $p = 2$, we have the following theorem.

Theorem 2.8. *Let $G \in BI(2)$. If $cl(G) = 3$, then $exp(G) = 8$ or $exp(G) = 4$.*

Proof. If $exp(G) \leqslant 2$, then G is abelian, which contradicts to $cl(G) = 3$. So we have $exp(G) \geqslant 4$.

Suppose that $exp(G) = 2^n > 8$. Let $a \in G$ such that $|a| = 2^n$. Let $H = \langle a, b\rangle$, where $b \in G$. Suppose that c be an element of minimal order such that $H = \langle a, c\rangle$. By Lemma 2.7, $\langle a\rangle \cap \langle c\rangle = 1$ and $|H'| \leqslant 2$. Then $|[a, b]| = |[a, c]| \leqslant 2$.

We claim that $[a, b] \in Z(G)$. It suffices to consider the case that $|[a, b]| = |[a, c]| = 2$. If $|c| = 2$, then $|\langle c\rangle^G| \leqslant 4$. Since $[a, c] \in \langle c\rangle^G$, $[a, c] \in Z(G)$. Now we consider $|c| \geqslant 4$. If $[a, c] \in \langle a\rangle$ or $[a, c] \in \langle c\rangle$, then $[a, c] \in Z(G)$ by Lemma 2.6. So we assume $[a, c] \notin \langle a\rangle$ and $[a, c] \notin \langle c\rangle$. Since H is also a $BI(2)$ 2-group, by Lemma 2.7, $[a, c] \in Z(H)$. Hence $\langle a\rangle^H = \langle a\rangle \times \langle [a, c]\rangle$ and $\langle c\rangle^H = \langle c\rangle \times \langle [a, c]\rangle$. Obviously, $\langle a\rangle^G = \langle a\rangle^H$ and $\langle c\rangle^G = \langle c\rangle^H$. Thus $H = \langle a\rangle^H \langle c\rangle^H \lhd G$, and $\langle [a, c]\rangle = H' \lhd G$. Since $|[a, c]| = 2$, we see $[a, c] \in Z(G)$.

Therefore $[x, y] \in Z(G)$ if $|x| = 2^n$ or $|y| = 2^n$. Now we consider the case that $|x|, |y| < 2^n$. By Hall–Petrescu formula,

$$(ax)^{2^{n-1}} = a^{2^{n-1}} x^{2^{n-1}} c_2^{2^{n-1}(2^{n-1}-1)/2} c_3^{2^{n-1}(2^{n-1}-1)(2^{n-1}-2)/6},$$

where $c_2 \in \gamma_2(\langle a, x\rangle)$, $c_3 \in \gamma_3(\langle a, x\rangle)$. By [5, Proposition 12], we see $(ax)^{2^{n-1}} = a^{2^{n-1}} x^{2^{n-1}} = a^{2^{n-1}}$, which implies $|ax| = 2^n$. Then $[ax, y] \in Z(G)$. Since $[ax, y] = [a, y]^x[x, y]$, it is clear that $[x, y] \in Z(G)$. Hence $G' \leqslant Z(G)$, which contradicts to $cl(G) = 3$. Therefore $exp(G) \leqslant 8$. □

Using the method in the proof of this theorem, we can get the following corollary.

Corollary 2.9. *Let G be a nonabelian $BI(2)$-group. If $exp(G) \geqslant 2^4$, then $cl(G) = 2$ and G' is an elementary abelian 2-group.*

We give two examples satisfying the requirement of this theorem.

Example 2.10. Let $G = \langle a, b \mid a^2 = b^2,\ a^4 = b^4 = c^2 = 1,\ [a, b] = a^2c,\ [a, c] = 1,\ [b, c] = a^2\rangle$. Then $G \in BI(2)$ with $exp(G) = 4$ and $cl(G) = 3$.

Example 2.11. Let $G = \langle a, b \mid a^8 = b^4 = 1,\ a^4 = b^2,\ a^b = b^{-1}\rangle$. Then G is a generalized quaternion group of order 16. It is easy to see $G \in BI(2)$ with $exp(G) = 8$ and $cl(G) = 3$.

3. $BI(p^2)$-groups with $p \geqslant 3$

Throughout this section, p will be a prime $\geqslant 3$. We first consider a $BI(p^2)$-group with a cyclic subgroup of index p^2. Finite p-groups with a cyclic subgroup of index p^2 were studied in [7], where these groups are given in terms of generators and relations. We use the classification of these groups in [1, Theorem 74.1].

Lemma 3.1. *If $|G| = p^{m+2}$ and $exp(G) = p^m$ with $m \geqslant 3$, then $|G'| \leqslant p^2$.*

Proof. By [1, Theorem 74.1], G is a metacyclic p-group or $|\Omega_1(G)| = p^3$ and $exp(\Omega_1(G)) = p$. If $|\Omega_1(G)| = p^3$ and $exp(\Omega_1(G)) = p$, let $a \in G$ such that $|a| = p^m$. Then $G = \langle a\rangle\Omega_1(G)$. Hence $G' \leqslant \Omega_1(G)$, and then $|G'| \leqslant p^2$.

Now we consider the case that G is a metacyclic p-group. Let R be a normal elementary abelian subgroup of order p^2 in G and $a \in G$ of order p^m. Since $a^{p^{m-1}}$ centralizes R and G is metacyclic, we have $|\langle a\rangle \cap R| = p$ and so $(R\langle a\rangle)/R$ is a cyclic subgroup of index p in G/R and G/R is noncyclic. Therefore there is $b \in G - (R\langle a\rangle)$ with $b^p \in R$ so that $|b| \leqslant p^2$. Also, $G/(R\langle a^p\rangle)$ is elementary abelian of order p^2 which gives $R\langle a^p\rangle = \Phi(G)$ and so $G = \langle a, b\rangle$. Since G is regular, $exp(\langle b\rangle^G) = p^2$. Then $exp(G') \leqslant p^2$ for $G' \leqslant \langle b\rangle^G$. Note that G' is cyclic for G is metacyclic, we have $|G'| \leqslant p^2$. □

Lemma 3.2. *Let $G = \langle x, y\rangle \in BI(p^2)$. If $\langle x\rangle \cap \langle y\rangle = 1$, then the following holds*:

(1) $cl(G) \leqslant 4$. *In particular*, $cl(G) \leqslant 3$ *if* $|x| \geqslant |y| \geqslant p^3$.
(2) $|G'| \leqslant p^3$, *and* $exp(G') \leqslant p^2$.
(3) $G^{p^2} \leqslant Z(G)$.

Proof. Let $|x| = p^m$, $|y| = p^n$. Since G is a $BI(p^2)$-group, $|\langle x\rangle^G : \langle x\rangle| \leqslant p^2$. Then $|\langle x\rangle||\langle x^y\rangle|/|\langle x\rangle \cap \langle x^y\rangle| = |\langle x\rangle\langle x^y\rangle| \leqslant |\langle x\rangle^G| \leqslant p^{m+2}$. We have $|\langle x\rangle \cap \langle x^y\rangle| \geqslant p^{m-2}$, and $x^{p^2} \in \langle x\rangle \cap \langle x^y\rangle$. Hence $\langle x^{p^2}\rangle = \langle (x^y)^{p^2}\rangle = \langle x^{p^2}\rangle^y$. Let $H = \langle x^{p^2}, y\rangle$. Then $\langle x^{p^2}\rangle \lhd H$. By Lemma 2.5, $x^{p^2} \in N_G(\langle y\rangle)$, and $\langle y\rangle \lhd H$. Since $\langle x\rangle \cap \langle y\rangle = 1$, $[x^{p^2}, y] = 1$, and $x^{p^2} \in Z(G)$. Similarly, $y^{p^2} \in Z(G)$.

Let $H = \langle x\rangle^G$, $K = \langle y\rangle^G$. Then $G' \leqslant H \cap K$. If $|H \cap K| \leqslant p^3$, then $cl(G) \leqslant 4$. Hence if $|y| \leqslant p^2$, then $|K| \leqslant p^4$ and $H \cap K < K$. Thus we also have $cl(G) \leqslant 4$. So we need only to consider the case that $|x|, |y| \geqslant p^3$. Without loss of generality, we assume that $m \geqslant n \geqslant 3$. We shall prove that $cl(G) \leqslant 3$ under this assumption, which completes the proof of (1). Since G is a $BI(p^2)$ p-group and $\langle x\rangle \cap \langle y\rangle = 1$, we see $|H \cap K| \leqslant p^4$. If $|H \cap K| = p^3$, then $x^{p^{m-1}} \in H \cap K$ and $y^{p^{n-1}} \in H \cap K$. As we have proved $x^{p^{m-1}}, y^{p^{n-1}} \in Z(G)$, we see $H \cap K \leqslant Z_2(G)$, which implies $cl(G) \leqslant 3$. So we need to consider the case that $|H \cap K| = p^4$. Hence $x^{p^{m-2}}, y^{p^{n-2}} \in H \cap K$. Since $\langle x\rangle \cap \langle y\rangle = 1$, $H \cap K = \langle x^{p^{m-2}}, y^{p^{n-2}}\rangle$. Let $A = \langle x^{p^{m-1}}, y^{p^{n-1}}\rangle$. By the above argument, $A \leqslant Z(G)$ for $m, n \geqslant 3$. Now consider $\bar{G} = G/A$. Then $\bar{G} = \langle \bar{x}, \bar{y}\rangle$, where $\bar{x} = xA$, $\bar{y} = yA$. Thus

$$\left|\langle \bar{x}\rangle^{\bar{G}} : \langle \bar{x}\rangle\right| = \left|HA/A : \langle x\rangle A/A\right| = \left|H : \langle x\rangle A\right| \leqslant p.$$

Similarly, $|\langle \bar{y}\rangle^{\bar{G}} : \langle \bar{y}\rangle| \leqslant p$. By Lemma 2.4, $|\bar{G}'| \leqslant p$, and then $|G'| \leqslant p^3$. If $|G'| \leqslant p^2$, then $cl(G) \leqslant 3$. If $|G'| = p^3$, then $A \leqslant G'$. As $A \leqslant Z(G)$, we also get $cl(G) \leqslant 3$.

Hence we have proved (1) and (2).

Let $g_1 = x^i y^j \in G$. By Hall–Petrescu formula,

$$\left(x^i y^j\right)^{p^2} = \left(x^i\right)^{p^2} \left(y^j\right)^{p^2} a_2^{l_2} a_3^{l_3} a_4^{l_4},$$

where $a_i \in \gamma_i(G)$, $l_i = (p^2)!/(p^2 - i)!i!$ for $i = 2, 3, 4$. If $p \geqslant 5$, then $p^2 \mid l_i$ for $i = 2, 3, 4$. Since $exp(G) \leqslant p^2$, then $g_1^{p^2} = (x^i y^j)^{p^2} = x^{ip^2} y^{jp^2} \in Z(G)$. For any $g = x^{k_1} y^{m_1} \cdots x^{k_r} y^{m_r} \in G$, we see that $g^{p^2} = (x^{k_1})^{p^2} (y^{m_1})^{p^2} \cdots (x^{k_r})^{p^2} (y^{m_r})^{p^2} \in Z(G)$.

Now we consider the case $p = 3$. We shall prove that $exp(\gamma_3(G)) = 3$ in the following cases.

(i) $|y| = 3$. Then $K = \langle y\rangle^G$ is of order $\leqslant 3^3$. Thus K is regular and generated by elements of order 3, and $exp(K) = 3$. Since $G' \leqslant K$, we see $exp(G') \leqslant 3$ and $exp(\gamma_3(G)) \leqslant 3$.

(ii) $|x| = |y| = 3^2$. Then $|H|, |K| \leqslant 3^4$. Hence $|H \cap K| \leqslant 3^3$. Obviously, $exp(\gamma_3(G)) \leqslant 3$ if $|H \cap K| \leqslant 3^2$. So we need only to consider that $|H \cap K| = 3^3$. Now $|H| = |K| = 3^4$, $|G| = 3^5$, and $|G'| \leqslant |H \cap K| = 3^3$. If $|G'| \leqslant 3^2$, then $exp(\gamma_3(G)) \leqslant 3$. Now consider that $|G'| = 3^3$. If G' is cyclic, then G is regular, and $exp(G) = 3^2$ for $|x| = |y| = 3^2$, a contradiction. Therefore $|G/\Omega_1(G')| \leqslant 3^3$, and $\gamma_3(G) \leqslant \Omega_1(G')$. Hence $exp(\gamma_3(G)) \leqslant 3$.

(iii) $|x| = 3^m \geqslant 3^3$ and $|y| = 3^2$. (Similarly for the case $|y| = 3^m \geqslant 3^3$ and $|x| = 3^2$.) As above, we need only to consider that $|H \cap K| = 3^3$. If $exp(H \cap K) \leqslant 3$, then $exp(\gamma_3(G)) \leqslant 3$. So we can assume $exp(H \cap K) \geqslant 3^2$. Let $B = \Omega_1(H \cap K)$. Since $x^{3^{m-1}}, y^3 \in B$, we have $|B| = 3^2$. Consider $\bar{G} = G/B$. Then $\bar{G} = \langle \bar{x}, \bar{y}\rangle$, where $\bar{x} = xB$, $\bar{y} = yB$. As above, Lemma 2.4 is applied to get that $|\bar{G}'| \leqslant 3$. Then $\gamma_3(G) \leqslant B$, and we see $exp(\gamma_3(G)) \leqslant 3$.

(iv) $|x|, |y| \geqslant 3^3$. As above $|G'| \leqslant 3^3$, and $\gamma_3(G) \leqslant \langle x^{3^{m-1}} \rangle \times \langle y^{3^{n-1}} \rangle$. Thus $exp(\gamma_3(G)) \leqslant 3$. Since $3^2 | l_2, l_4$, $3 | l_3$, and $exp(\gamma_3(G)) \leqslant 3$, then we see that $G^{3^2} \leqslant Z(G)$ by Hall–Petrescu formula as above. $\square$

Lemma 3.3. *Let $G = \langle x, y \rangle \in BI(p^2)$ be noncyclic, $|x| = p^m$, $|y| = p^n$. If $m \geqslant n$ and $m \geqslant 3$, then there exists y_1 such that $G = \langle x, y_1 \rangle$ and $\langle x \rangle \cap \langle y_1 \rangle = 1$, or $\langle x \rangle \cap \langle y_1 \rangle = \langle y_1^p \rangle$ and $|y_1| = p^2$, where $p = 3$.*

Proof. We proceed by induction on $m + n$. Let $H = \langle x \rangle^G$, $K = \langle y \rangle^G$.

We first prove the special cases that $n = 2$ and $m = n = 3$, which illustrates the idea in the general case.

(a) $n = 2$, and $\langle x \rangle \cap \langle y \rangle = \langle y^p \rangle$. If $p = 3$, then $y_1 = y$ satisfies all the requirements. Hence we need only to consider $p \geqslant 5$. Let $M = \langle x^{p^{m-2}}, y \rangle$. It is clear that $|M| \leqslant p^5$. Hence M is regular. Since $y^p = x^{dp^{m-1}}$ for some integer d by the assumption, we see $(yx^{-dp^{m-1}})^p = 1$ by the regularity of M. Let $y_1 = yx^{-dp^{m-1}}$. Then $G = \langle x, y_1 \rangle$ and $\langle x \rangle \cap \langle y_1 \rangle = 1$.

(b) $m = n = 3$. We shall prove that $cl(G) \leqslant 5$, $exp(G') \leqslant p^2$ and $exp(\gamma_3(G)) \leqslant p$ in the following two cases.

(b1) $\langle x \rangle \cap \langle y \rangle = \langle x^p \rangle$. Then $x^p \in Z(G)$, and $x^p = (x^y)^p$. Since G is a $BI(p^2)$ p-group, $|H| \leqslant p^5$. By [1, Theorem 74.1], H is metacyclic, or $|\Omega_1(H)| = p^3$ and $exp(\Omega_1(H)) = p$. If H is metacyclic, then H is a regular p-group. Since $x, x^y \in H$, we see that $(x^{-1}x^y)^p = 1$, and $exp(G') = p$ for $G' \leqslant H$. Hence $G' \leqslant \Omega_1(H)$. Since H is a metacyclic p-group, it follows that $|\Omega_1(H)| \leqslant p^2$, and then $cl(G) \leqslant 3$. So we need only to consider the case that $|\Omega_1(H)| = |\Omega_1(K)| = p^3$ and $exp(\Omega_1(H)) = exp(\Omega_1(K)) = p$. Let $\bar{G} = G/\Omega_1(H)$, $\bar{x} = x\Omega_1(H)$. Then $|\bar{x}| = p^2$ and $\langle \bar{x} \rangle \lhd \bar{G}$. Obviously $\bar{G}' \leqslant \langle \bar{x} \rangle$. Therefore $|\bar{G}'| \leqslant p$, and $\gamma_3(G) \leqslant \Omega_1(H)$. We have $exp(G') \leqslant p^2$, $exp(\gamma_3(G) \leqslant p$ and $cl(G) \leqslant 5$.

(b2) $\langle x \rangle \cap \langle y \rangle = \langle x^{p^2} \rangle$. If $|\Omega_1(H)| = p^3$ or $|\Omega_1(K)| = p^3$, the argument in (b1) can be applied to see that $exp(G') \leqslant p^2$, $exp(\gamma_3(G) \leqslant p$ and $cl(G) \leqslant 5$. So we need to consider the following two subcases.

(b2.1) H, K are metacyclic and $H \cap K$ is cyclic. Since $G' \leqslant H \cap K$, then $G' = \langle [x, y] \rangle$. Since $x, x^y \in H$ and $x^{p^2} = (x^{p^2})^y = (x^y)^{p^2}$, we see that $[x, y]^{p^2} = 1$ by the regularity of H. Hence $|G'| \leqslant p^2$, and $|\gamma_3(G)| \leqslant p$, $cl(G) \leqslant 3$.

(b2.2) H, K are metacyclic and $H \cap K$ is noncyclic. Since p is odd, $|\Omega_1(H \cap K)| \geqslant p^2$ and H and K are metacyclic, we have $|\Omega_1(H)| = |\Omega_1(K)| = p^2$. Hence $\Omega_1(H) = \Omega_1(H \cap K) = \Omega_1(K)$. Let $\bar{G} = G/\Omega_1(H)$, $\bar{x} = x\Omega_1(H)$, $\bar{y} = y\Omega_1(H)$. Then $\bar{G} = \langle \bar{x}, \bar{y} \rangle$, $\langle \bar{x} \rangle \cap \langle \bar{y} \rangle = 1$ and $|\langle \bar{x} \rangle^{\bar{G}} : \langle \bar{x} \rangle| \leqslant p$, $|\langle \bar{y} \rangle^{\bar{G}} : \langle \bar{y} \rangle| \leqslant p$. By Lemma 2.4, $|\bar{G}'| \leqslant p$, and $\gamma_3(G) \leqslant \Omega_1(H)$. Therefore $exp(G') \leqslant p^2$, $exp(\gamma_3(G)) \leqslant p$, and $cl(G) \leqslant 4$.

Now $\langle x \rangle \cap \langle y \rangle = \langle y^p \rangle$ or $\langle y^{p^2} \rangle$, $x^{p^2} = y^{kp^2}$, where $1 \leqslant k \leqslant p - 1$. Since $cl(G) \leqslant 5$, by Hall–Petrescu formula,

$$(xy^{-k})^{p^2} = x^{p^2} y^{-kp^2} c_2^{n_2} c_3^{n_3} c_4^{n_4} c_5^{n_5},$$

where $c_i \in \gamma_i(G)$, and $p^2 \mid n_2$, $p \mid n_3$, $p \mid n_4$, $p \mid n_5$ for $p \geqslant 3$. Hence $(xy^{-k})^{p^2} = 1$. Let $y_0 = xy^{-k}$, then $G = \langle x, y_0 \rangle$ and $|y_0| \leqslant p^2$. By (a), there exists y_1 such that $G = \langle x, y_1 \rangle$ and $\langle x \rangle \cap \langle y_1 \rangle = 1$, or $\langle x \rangle \cap \langle y_1 \rangle = \langle y_1^3 \rangle$ and $|y_1| = 9$.

(c) $m = n \geqslant 4$.

(c1) $\langle x^{p^2} \rangle \leqslant \langle x \rangle \cap \langle y \rangle$. Hence $x^{p^2} \in Z(G)$ and $\langle x^{p^3} \rangle \lhd G$. Consider $\bar{G} = G/\langle x^{p^3} \rangle = \langle \bar{x}, \bar{y} \rangle$, where $\bar{x} = x\langle x^{p^3} \rangle$, $\bar{y} = y\langle x^{p^3} \rangle$. Obviously, $|\bar{x}| = |\bar{y}| = p^3$. By the argument in (b), there exists $y_1 \in G$ such that $\bar{G} = \langle \bar{x}, \bar{y_1} \rangle$ and $|\bar{y_1}| \leqslant p^2$. Thus $G = \langle x, y_1 \rangle$ and $|y_1| < |x|$. Let $G_0 = \langle x^p, y_1 \rangle$. By induction, there exists y_2 such that $G_0 = \langle x^p, y_2 \rangle$, and $\langle x^p \rangle \cap \langle y_2 \rangle = 1$, or $\langle x^p \rangle \cap \langle y_2 \rangle = \langle y_2^p \rangle$ where $p = 3$. Therefore $G = \langle x, G_0 \rangle = \langle x, y_2 \rangle$ and $\langle x \rangle \cap \langle y_2 \rangle = 1$, or $\langle x \rangle \cap \langle y_2 \rangle = \langle y_2^p \rangle$ and $|y_2| = 9$.

(c2) $\langle x \rangle \cap \langle y \rangle = \langle x^{p^{m-1}} \rangle$.

We shall prove that $exp(G') \leqslant p^3$, $exp(\gamma_3(G)) \leqslant p^2$, $exp(\gamma_4(G)) \leqslant p$ and $cl(G) \leqslant 6$.

By [1, Theorem 74.1], H is a metacyclic group or $|\Omega_1(H)| = p^3$ and $exp(\Omega_1(H)) = p$. Similarly, K is metacyclic or $|\Omega_1(K)| = p^3$ and $exp(\Omega_1(K)) = p$.

(c2.1) If $|\Omega_1(H)| = p^3$ and $exp(\Omega_1(H)) = p$. Consider $\bar{G} = G/\Omega_1(H) = \langle \bar{x}, \bar{y} \rangle$, where $\bar{x} = x\Omega_1(H)$, $\bar{y} = y\Omega_1(H)$. Obviously, $\langle \bar{x} \rangle \cap \langle \bar{y} \rangle = 1$. By Lemma 3.2, $exp(\bar{G}') \leqslant p^2$. Since $|\bar{x}| = p^{m-1} = |\bar{H}|$, we have that $\langle\bar{x}\rangle = \bar{H} \lhd \bar{G}$. Then $\bar{G}'$ is a cyclic group of order $\leqslant p^2$ for $\bar{G}' \leqslant \bar{H} = \langle\bar{x}\rangle$. Therefore $exp(G') \leqslant p^3$, $exp(\gamma_3(G)) \leqslant p^2$ and $exp(\gamma_4(G)) \leqslant p$. Since $|G'| \leqslant p^5$, $cl(G) \leqslant 6$.

(c2.2) H, K are metacyclic and $H \cap K$ is cyclic. Since $G' \leqslant H \cap K$, G' is cyclic, and $G' = \langle [x, y]\rangle$. Let $\bar{G} = G/\langle x^{p^{m-1}}\rangle = \langle\bar{x}, \bar{y}\rangle$, where $\bar{x} = x\langle x^{p^{m-1}}\rangle$, $\bar{y} = y\langle x^{p^{m-1}}\rangle$. Then $\langle\bar{x}\rangle \cap \langle\bar{y}\rangle = 1$. By Lemma 3.2, $|[\bar{x}, \bar{y}]| \leqslant p^2$, and $|[x, y]| \leqslant p^3$. Hence $exp(G') \leqslant p^3$, $exp(\gamma_3(G)) \leqslant p^2$, $exp(\gamma_4(G)) \leqslant p$ and $cl(G) \leqslant 4$.

(c2.3) H, K are metacyclic and $H \cap K$ is noncyclic. By above argument in (b2.2), $\Omega_1(H) = \Omega_1(K)$ are of order p^2. Consider $\bar{G} = G/\Omega_1(H) = \langle\bar{x}, \bar{y}\rangle$, where $\bar{x} = x\Omega_1(H), \bar{y} = y\Omega_1 H$. It is easy to see that $|\langle\bar{x}\rangle^{\bar{G}} : \langle\bar{x}\rangle| \leqslant p$, $|\langle\bar{y}\rangle^{\bar{G}} : \langle\bar{y}\rangle| \leqslant p$ and $\langle\bar{x}\rangle \cap \langle\bar{y}\rangle = 1$. By Lemma 2.4, $|\bar{G}'| \leqslant p$. Hence $|G'| \leqslant p^3$, and $exp(G') \leqslant p^3$, $exp(\gamma_3(G)) \leqslant p^2$, $exp(\gamma_4(G)) \leqslant p$ and $cl(G) \leqslant 4$.

Since $\langle x\rangle \cap \langle y\rangle = \langle x^{p^{m-1}}\rangle$, $x^{p^{m-1}} = y^{-lp^{m-1}}$ for some positive integer $l \leqslant p-1$. By Hall–Petrescu formula,

$$\left(xy^l\right)^{p^{m-1}} = x^{p^{m-1}} y^{lp^{m-1}} c_2^{n_2} c_3^{n_3} c_4^{n_4} c_5^{n_5} c_6^{n_6},$$

where $c_i \in \gamma_i(G)$, $n_i = (p^{m-1})!/(p^{m-1}-i)!i!$ for $i = 2, 3, \dots, 6$. Since $p \geqslant 3$, we see that $p^3 \mid n_2$, $p^2 \mid n_i$ for $i = 3, \dots, 6$. Hence $(xy^l)^{p^{m-1}} = 1$. Therefore $G = \langle x, xy^l\rangle$ such that $|x| > |xy^l|$. By induction, there exists y_1 such that $G = \langle x, y_1\rangle$ and $\langle x\rangle \cap \langle y_1\rangle = 1$, or $\langle x\rangle \cap \langle y_1\rangle = \langle y_1^3\rangle$ and $|y_1| = 9$.

(c3) $\langle x\rangle \cap \langle y\rangle = \langle x^{p^{m-k}}\rangle$ for some $k \geqslant 2$. In this case, we consider the quotient group $\bar{G} = G/\langle x^{p^{m-k-1}}\rangle = \langle\bar{x}, \bar{y}\rangle$, where $\bar{x} = x\langle x^{p^{m-k-1}}\rangle$, $\bar{y} = y\langle x^{p^{m-k-1}}\rangle$. Then (c2) is applied to find $y_0 \in G$ such that $\bar{G} = \langle\bar{x}, \bar{y}_0\rangle$ with $|\bar{y}_0| < |\bar{y}|$. Hence $G = \langle x, y_0\rangle$ with $|y_0| < |y|$. By induction, there exits y_1 satisfying all the requirements.

(d) $m > n \geqslant 3$. Let $G_1 = \langle x^{p^{m-n}}, y\rangle$. By induction, there exists y_1 such that $G_1 = \langle x^{p^{m-n}}, y_1\rangle$ and $\langle x^{p^{m-n}}\rangle \cap \langle y_1\rangle = 1$, or $\langle x^{p^{m-n}}\rangle \cap \langle y_1\rangle = \langle y_1^3\rangle$ and $|y_1| = 9$. Hence $G = \langle x, G_1\rangle = \langle x, y_1\rangle$ satisfying all the requirements. $\square$

Theorem 3.4. *Let $G \in BI(p^2)$. Then $G^{p^2} \leqslant Z(G)$ and $exp(G') \leqslant p^2$.*

Proof. We first prove that $G^{p^2} \leqslant Z(G)$. Let $x \in G$ such that $|x| = p^m \geqslant p^3$. For any $y \in G$, we consider $G_1 = \langle x, y\rangle$. In order to prove $G^{p^2} \leqslant Z(G)$, it suffices to prove $[x^{p^2}, y] = 1$. If $\langle x\rangle \cap \langle y\rangle = 1$, by Lemma 3.2, $G_1^{p^2} \leqslant Z(G_1)$, which implies $[x^{p^2}, y] = [x, y^{p^2}] = 1$.

(i) $|y| \leqslant p^2$. By above argument, we need only to consider that $|y| = p^2$ and $\langle x\rangle \cap \langle y\rangle = \langle y^p\rangle$. If $m = 3$, then $x^{p^2} \in \langle y^p\rangle \leqslant Z(G_1)$ and so $[x^{p^2}, y] = 1$. Let $m > 3$ and $x_0 \in \langle x\rangle$ be such that $x_0^p = y^{-p}$. By Lemma 2.5, $|\langle x\rangle : N_{\langle x\rangle}(\langle y\rangle)| \leqslant p^2$ and so $\langle x_0\rangle$ normalizes $\langle y\rangle$ and therefore $[x_0, y] \in \langle y^p\rangle$ so that $\langle x_0, y\rangle$ is of order p^3 and class $\leqslant 2$. Set $y_1 = x_0 y$ so that $y_1^p = x_0^p y^p = 1$ and $G_1 = \langle x, y\rangle = \langle x, y_1\rangle$. By Lemma 3.2, $x^{p^2} \in Z(G_1)$ and we are done.

(ii) $|y| \geqslant p^3$. Without loss of generality, assume that $|x| \geqslant |y|$. If there exists y_1 such that $G_1 = \langle x, y\rangle = \langle x, y_1\rangle$ and $\langle x\rangle \cap \langle y_1\rangle = 1$, by Lemma 3.2, we have that $G_1^{p^2} \leqslant Z(G_1)$, which means $[x^{p^2}, y] = [x, y^{p^2}] = 1$. By Lemma 3.3, there is only one exceptional case that $p = 3$, $G_1 = \langle x, y\rangle = \langle x, y_1\rangle$, $\langle x\rangle \cap \langle y_1\rangle = \langle y_1^3\rangle$ and $|y_1| = 9$. By (i) applied to $G_1 = \langle x, y_1\rangle$, we get $x^{3^2} \in Z(G_1)$.

Next we prove $exp(G') \leqslant p^2$.

We claim that $|[x, y]| \leqslant p^2$ for all $x, y \in G$. If $|y| \leqslant p^2$, then $|\langle y\rangle^G| \leqslant p^4$. Suppose that $exp(\langle y\rangle^G) \geqslant p^3$. If $exp(\langle y\rangle^G) = p^4$, $\langle y\rangle^G$ is a normal cyclic subgroup of G, and then $\langle y\rangle \lhd G$, a contradiction. Hence $exp(\langle y\rangle^G) = p^3$, and there exists $w \in \langle y\rangle^G$ such that $|w| = p^3$. This implies $\langle y\rangle^G$ is regular, and $exp(\langle y\rangle^G) = |y| \leqslant p^2$, also a contradiction. Therefore $exp(\langle y\rangle^G) \leqslant p^2$. Hence $|[x, y]| \leqslant p^2$ for $[x, y] \in \langle y\rangle^G$ for every $x \in G$. If $|x| \geqslant |y| \geqslant p^3$, consider that $G_1 = \langle x, y\rangle$. By Lemma 3.3, there exists y_1 such that $G_1 = \langle x, y\rangle = \langle x, y_1\rangle$ and $\langle x\rangle \cap \langle y_1\rangle = 1$, or $\langle x\rangle \cap \langle y_1\rangle = \langle y_1^3\rangle$ and $|y_1| = 9$. If $\langle x\rangle \cap \langle y_1\rangle = 1$, by Lemma 3.2, $|[x, y]| \leqslant p^2$. If $\langle x\rangle \cap \langle y_1\rangle \neq 1$, then $|y_1| = 3^2$, we also see that $|[x, y]| \leqslant 3^2$ by the above argument.

Now it suffices to prove that $exp(\langle a, b\rangle) \leqslant p^2$ for any $a, b \in G$ with $|a|, |b| \leqslant p^2$. Let $A = \langle a, b\rangle$, $H = \langle a\rangle^A$, $K = \langle b\rangle^A$. Hence $|H|, |K| \leqslant p^4$ and $|A| \leqslant p^6$. If $|A| = p^6$, it is easy to see that $|H \cap K| \leqslant p^2$, and then $|A'| \leqslant p^2$, $|\gamma_3(A)| \leqslant p$ and $cl(A) \leqslant 3$. By Hall–Petrescu formula, $exp(A) \leqslant p^2$. So we need to consider $|A| \leqslant p^5$. Suppose $exp(A) \geqslant p^3$. By [1, Theorem 74.1], A is metacyclic, or $|\Omega_1(A)| = p^3$ and $exp(\Omega_1(A)) = p$. If A is metacyclic, then A is regular, and $exp(A) \leqslant p^2$. Now consider the case that $|\Omega_1(A)| = p^3$ and $exp(\Omega_1(A)) = p$. Hence $exp(\Omega_2(A)) \leqslant p^2$. But $a, b \in \Omega_2(A)$, a contradiction. Hence $exp(A) \leqslant p^2$, and then $exp(G') \leqslant p^2$. □

Theorem 3.5. *Let $G \in BI(p^2)$. Then $cl(G) \leqslant 4$.*

Proof. It suffices to prove that $[x, y] \in Z_3(G)$ for any $x, y \in G$. Let $|x| = p^m$, $|y| = p^n$, $H = \langle x\rangle^G$, $K = \langle y\rangle^G$. If $|H \cap K| \leqslant p^3$, then $[x, y] \in Z_3(G)$ for $[x, y] \in H \cap K \lhd G$.

(i) $m \leqslant 2$ and $n \leqslant 2$. Hence $|H|, |K| \leqslant p^4$, and $|H'|, |K'| \leqslant p^2$. If $y \in H$, then $[x, y] \in H'$, and $[x, y] \in Z_2(G) \leqslant Z_3(G)$. Similarly, $[x, y] \in Z_3(G)$ if $x \in K$. Now assume $x \notin K$, $y \notin H$. Thus $|H \cap K| \leqslant p^3$, and then $[x, y] \in Z_3(G)$.

(ii) $m \leqslant 2$ and $n \geqslant 3$. (Similarly for $m \geqslant 3$ and $n \leqslant 2$.) Since $|H| \leqslant p^4$, we need only to consider $|H \cap K| = p^4$. Hence $H \leqslant K$. By Lemma 3.1, $|K'| \leqslant p^2$. Therefore $[x, y] \in Z_3(G)$.

(iii) $m \geqslant 3$ and $n \geqslant 3$. If $\langle x\rangle \cap \langle y\rangle = 1$, then $|H \cap K| \leqslant p^4$. As before, we need only to consider that $|H \cap K| = p^4$. Thus $x^{p^{m-1}} \in \langle x\rangle \cap K$, and $x^{p^{m-1}} \in H \cap K$. Similarly, $y^{p^{n-1}} \in H \cap K$. By Theorem 3.4, $x^{p^{m-1}}, y^{p^{n-1}} \in Z(G)$. Hence $H \cap K \leqslant Z_3(G)$, and $[x, y] \in Z_3(G)$. Now we consider that $\langle x\rangle \cap \langle y\rangle \neq 1$. Let $G_1 = \langle x, y\rangle$. We can assume $m \geqslant n$. By Lemma 3.3, there exists y_1 such that $G_1 = \langle x, y\rangle = \langle x, y_1\rangle$ and $\langle x\rangle \cap \langle y_1\rangle = 1$, or $\langle x\rangle \cap \langle y_1\rangle = \langle y_1^3\rangle$ and $|y_1| = 9$. If $\langle x\rangle \cap \langle y_1\rangle = 1$, we see $G_1' \leqslant Z_3(G)$ as above, and then $[x, y] \in Z_3(G)$. In the case that $\langle x\rangle \cap \langle y_1\rangle \neq 1$, by (ii) above, we also have $[x, y] \in G_1' \leqslant Z_3(G)$ for $|y_1| = 9$. □

Lemma 3.6. *Let $G \in BI(p^2)$. Suppose $x, y \in G$, $|x| = p^m \geqslant p^3$ and $|y| = p^n \geqslant p^3$. If $\langle x\rangle \cap \langle y\rangle = 1$, then $[x, y] \in Z_2(G)$.*

Proof. By Theorem 3.4, $x^{p^{m-1}}, y^{p^{n-1}} \in Z(G)$. Let $H = \langle x\rangle^G$, $K = \langle y\rangle^G$. Since G is a $BI(p^2)$-group, $|H \cap K| \leqslant p^4$. If $|H \cap K| \leqslant p^2$, then $[x, y] \in H \cap K \leqslant Z_2(G)$. If $|H \cap K| = p^3$, then $x^{p^{m-1}}, y^{p^{n-1}} \in Z(G)$, and $[x, y] \in H \cap K \leqslant Z_2(G)$. So we need only to consider $|H \cap K| = p^4$. If $m \geqslant 4$, by Theorem 3.4, $x^{p^{m-2}} \in Z(G)$. Since $y^{p^{n-1}} \in Z(G)$ and $x^{p^{m-2}}, y^{p^{n-1}} \in H \cap K$, we have that $[x, y] \in H \cap K \leqslant Z_2(G)$. Similarly for the case that $n \geqslant 4$.

Now assume that $|H \cap K| = p^4$ and $m = n = 3$. Since $\langle x\rangle \cap \langle y\rangle = 1$ we have $|H \cap \langle y\rangle| = p^2$ and so $y^p \in H$. Thus $H = \langle x, y^p\rangle = \langle x\rangle\langle y^p\rangle$ is of order p^5 and $HK \lhd G$ is of order p^6 since $HK = H\langle y\rangle$ with $y^p \in H$. But $|\langle x\rangle\langle y\rangle| = |\langle x\rangle||\langle y\rangle| = p^6$ and so $HK = \langle x\rangle\langle y\rangle$. By [6, Satz 11.5, Kapitel III], HK is metacyclic and so regular. Thus $exp(HK) = p^3$ which implies that $(HK)' \lhd G$ is cyclic of order $\leqslant p^2$. Hence $[x, y] \in (HK)' \leqslant Z_2(G)$. □

Theorem 3.7. *Let $G \in BI(p^2)$. Then $cl(G) \leqslant 3$ if and only if $[\Omega_2(G), G] \leqslant Z_2(G)$.*

Proof. Obviously, we need only to prove that our condition is sufficient. Suppose $x, y \in G$. If $|y| \leqslant p^2$ (or $|x| \leqslant p^2$), then $[x, y] \in Z_2(G)$ by assumption. Hence we can assume that $|x| \geqslant |y| \geqslant p^3$. Let $G_1 = \langle x, y\rangle$. By Lemma 3.3, there exists y_1 such that $G_1 = \langle x, y_1\rangle$ and $\langle x\rangle \cap \langle y_1\rangle = 1$, or $\langle x\rangle \cap \langle y_1\rangle = \langle y_1^3\rangle$ and $|y_1| = 9$. If $|y_1| \leqslant p^2$, then $[x, y_1] \in Z_2(G)$ by assumption, and $[x, y] \in G_1' \leqslant Z_2(G)$. If $|y_1| \geqslant p^3$, then $\langle x\rangle \cap \langle y_1\rangle = 1$, by Lemma 3.6, $[x, y_1] \in Z_2(G)$, and $[x, y] \in G_1' \leqslant Z_2(G)$. Therefore $cl(G) \leqslant 3$. □

Theorem 3.8. *Let $G \in BI(p^2)$. If $cl(G) = 4$, then G^{p^2} is cyclic.*

Proof. Suppose that G^{p^2} is not cyclic. Then we need only to get a contradiction for the case that $exp(G) = p^m \geqslant p^3$.

Let $x \in G$ such that $|x| = p^m = exp(G)$. We shall prove that $[x, y] \in Z_2(G)$ for every $y \in G$. Let $G_1 = \langle x, y\rangle$. By Lemma 3.3, there exists $y_1 \in G_1$ such that $G_1 = \langle x, y_1\rangle$ and $\langle x\rangle \cap \langle y_1\rangle = 1$, or $\langle x\rangle \cap \langle y_1\rangle = \langle y_1^3\rangle$

and $|y_1| = 3^2$. If $|y_1| \geqslant p^3$, then $\langle x\rangle \cap \langle y_1\rangle = 1$. By Lemma 3.6, we have $[x, y_1] \in Z_2(G)$ and so $[x, y] \in Z_2(G)$. Now consider the case $|y_1| \leqslant p^2$, which include the case that $p = 3$, $\langle x\rangle \cap \langle y_1\rangle = \langle y_1^3\rangle$ and $|y_1| = 3^2$. We claim that there exists $z \in G$ such that $z^{p^2} \notin \langle x\rangle$. Otherwise, $g^{p^2} \in \langle x\rangle$ for every $g \in G$. Since $|x| = p^m = exp(G)$, then we have $g^{p^2} \in \langle x^{p^2}\rangle$, and then $G^{p^2} = \langle x^{p^2}\rangle$ is cyclic, a contradiction. Let $G_2 = \langle x, z\rangle$. By Lemma 3.3, there exists $z_1 \in G_2$ such that $G_2 = \langle x, z_1\rangle$ and $\langle x\rangle \cap \langle z_1\rangle = 1$, or $\langle x\rangle \cap \langle z_1\rangle = \langle z_1^3\rangle$ and $|z_1| = 3^2$. If $|z_1| \leqslant p^2$, we have that $|G_2'| \leqslant p^3$ for G_2' is a proper subgroup of $\langle z_1\rangle^{G_2}$ and $G_2 \in BI(p^2)$. By Theorem 3.4, $exp(G_2') \leqslant p^2$. If $|G_2'| = p^3$, then $|\Omega_1(G_2')| \geqslant p^2$, and $\gamma_3(G_2) \leqslant \Omega_1(G_2')$. Thus we have that $exp(\gamma_3(G_2)) \leqslant p$. By Hall–Petrescu formula, we see that $(xz_1)^{p^2} = x^{p^2} z_1^{p^2} = x^{p^2}$. In the way, we see that $g_2^{p^2} \in \langle x^{p^2}\rangle$ for every $g_2 \in G_2$, which contradicts to the choice of z. Hence we have $|z_1| \geqslant p^3$, and then $\langle x\rangle \cap \langle z_1\rangle = 1$. By Lemma 3.6, we have $[x, z_1] \in Z_2(G)$. Let $a = z_1 y_1$. Since $|y_1| \leqslant p^2$, using the above method for $|z_1| \leqslant p^2$, we see that $a^{p^2} = z_1^{p^2} y_1^{p^2} = z_1^{p^2}$. It follows that $|a| = |z_1|$ and $\langle x\rangle \cap \langle a\rangle = 1$. By Lemma 3.6, we have $[x, a] \in Z_2(G)$. Since $[x, a] = [x, z_1][x, y_1]^{z_1}$, we have $[x, y_1]^{z_1} = [x, z_1]^{-1}[x, a] \in Z_2(G)$, and $[x, y] \in Z_2(G)$.

Next we shall prove that $[x_1, x_2] \in Z_2(G)$ for every $x_1, x_2 \in G$, which contradicts to the assumption $cl(G) = 4$. If $|x_1| = p^m$ or $|x_2| = p^m$, $[x_1, x_2] \in Z_2(G)$ by the above argument. So we need only to consider the case that $|x_1| < p^m$ and $|x_2| < p^m$. Let $G_3 = \langle x, x_1\rangle$. By Lemma 3.3, there exists $x_1' \in G_3$ such that $G_3 = \langle x, x_1'\rangle$ and $\langle x\rangle \cap \langle x_1'\rangle = 1$, or $\langle x\rangle \cap \langle x_1'\rangle = \langle x_1'^3\rangle$ with $|x_1'| = 9$. If $\langle x\rangle \cap \langle x_1'\rangle = 1$, by Lemma 3.2, we have $|G_3'| \leqslant p^3$. If $|x_1'| = 3^2$, by the above argument in last paragraph, we also have $|G_3'| \leqslant 3^3$. Hence, as above, we see that $\gamma_3(G_3) \leqslant \Omega_1(G_3')$. By Hall–Petrescu formula, we have that $(xx_1)^{p^{m-1}} = x^{p^{m-1}} x_1^{p^{m-1}} = x^{p^{m-1}}$, and $|xx_1| = p^m$. Thus we have $[xx_1, x_2] \in Z_2(G)$. As we have proved that $[x, x_2] \in Z_2(G)$, we see that $[x_1, x_2] \in Z_2(G)$, as required. □

4. Some special $BI(p^m)$-groups

Every finite p-group is a $BI(p^m)$-group for some m. It is difficult to study $BI(p^m)$-groups without any restriction. In this section, only some special $BI(p^m)$-groups are considered.

Theorem 4.1. *Let $G \in BI(p^m)$ with $p \geqslant 3$. If G is regular, then $G^{p^m} \leqslant Z(G)$, $cl(G) < 1 + (2mp - 1)/(p - 1)$, and $|G'| \leqslant p^{(m+1)(2m+1)}$. If, in addition, $exp(G) = p^r \leqslant p^m$, then $cl(G) < 1 + (p(m + r - 1) - 1)/(p - 1)$ and $|G'| \leqslant p^{(m+r)(m+r-1)/2}$.*

Proof. Let $x \in G$ with $|x| = p^n \geqslant p^m$. For every $y \in G$, we consider $G_1 = \langle x, y\rangle$. We shall prove $G_1^{p^m} \leqslant Z(G_1)$, and then $x^{p^m} \in Z(G)$ and $G^{p^m} \leqslant Z(G)$. Without loss of generality, assume that $|x| \geqslant |y|$. If $\langle x\rangle \cap \langle y\rangle \neq 1$, then $x^{lp^r} = y^{p^s}$ for some integer l, r, s, and $(l, p) = 1$, $|y| > p^s$. Since $|x| \geqslant |y|$, $r \geqslant s$. Let $y_1 = x^{-lp^{r-s}} y$. Since G is regular, $y_1^{p^s} = 1$, and $|y_1| < |y|$. Obviously, $G_1 = \langle x, y_1\rangle$. In this way, we can find z such that $G_1 = \langle x, z\rangle$ with $\langle x\rangle \cap \langle z\rangle = 1$. As G_1 is also a $BI(p^m)$ group, $|\langle x\rangle^{G_1} : \langle x\rangle| \leqslant p^m$, and $|\langle x\rangle \cap \langle x^z\rangle| \geqslant p^{n-m}$. Thus $\langle x^{p^m}\rangle, \langle (x^z)^{p^m}\rangle \leqslant \langle x\rangle \cap \langle x^z\rangle$, and $\langle x^{p^m}\rangle = \langle (x^z)^{p^m}\rangle = \langle x^{p^m}\rangle^z$. Let $M = \langle x^{p^m}, z\rangle$. Then $\langle x^{p^m}\rangle \lhd M$. By Lemma 2.5, $\langle z\rangle \lhd M$. Hence $[x^{p^m}, z] = 1$ for $\langle x\rangle \cap \langle z\rangle = 1$, and then $x^{p^m} \in Z(G_1)$. Similarly $z^{p^m} \in Z(G_1)$. We see that $G_1^{p^m} \leqslant Z(G_1)$ for G_1 is regular.

For any $a \in G$, $N_G(\langle a\rangle)/C_G(\langle a\rangle)$ is isomorphic to subgroup of $Aut(\langle a\rangle)$. Since $p \geqslant 3$, $N_G(\langle a\rangle)/C_G(\langle a\rangle)$ is cyclic. Since $G^{p^m} \leqslant Z(G)$, we see that $exp(N_G(\langle a\rangle)/C_G(\langle a\rangle)) \leqslant p^m$. Hence $|N_G(\langle a\rangle)/C_G(\langle a\rangle)| \leqslant p^m$. By Lemma 2.5, $|G : N_G(\langle a\rangle)| \leqslant p^m$. We have

$$\left|G : C_G\left(\langle a\rangle\right)\right| = \left|G : N_G\left(\langle a\rangle\right)\right|\left|N_G\left(\langle a\rangle\right) : C_G\left(\langle a\rangle\right)\right| \leqslant p^{2m}.$$

This means that $b(G) \leqslant 2m$. By [2, Theorem 3], $cl(G) < 1 + (2mp - 1)/(p - 1)$. By the main theorem in [8], $|G'| \leqslant p^{(m+1)(2m+1)}$.

Now suppose $exp(G) = p^r \leqslant p^m$. Then $|N_G(\langle b\rangle) : C_G(\langle b\rangle)| \leqslant p^{r-1}$ for every $b \in G$. Then we see that $b(G) \leqslant m + r - 1$. By the same results above, we see that $cl(G) < 1 + (p(m + r - 1) - 1)/(p - 1)$ and $|G'| \leqslant p^{(m+r)(m+r-1)/2}$. □

The following example implies that the order of G' in Theorem 4.1 is best possible when $r = 1$.

Example 4.2. Let $G = \langle x_1, \ldots, x_{m+1}\rangle$, where $[x_i, x_j] = a_{ij}$, $a_{ij} \in Z(G)$, $|x_i| = |a_{jk}| = p$, $i = 1, \ldots, m+1$, $1 \leqslant j < k \leqslant m+1$. It is easy to see that $G \in BI(p^m)$ with $cl(G) = 2$, and $G' = Z(G) = \langle a_{12}\rangle \times \cdots \times \langle a_{i,i+1}\rangle \times \cdots \times \langle a_{m,m+1}\rangle$, and then $|G'| = p^{m(m+1)/2}$.

Theorem 4.3. *Let $G \in BI(p^m)$ with $p \geqslant 3$. If $exp(G) = p$ and $cl(G) = m+1$, then $|G'| = p^m$.*

Proof. Since $cl(G) = m+1$, there exist x, y such that $[x, y] \in Z_m(G) - Z_{m-1}(G)$. Hence $|\langle [x, y]\rangle^G| \geqslant p^m$. Since $|\langle x\rangle^G : \langle x\rangle| \leqslant p^m$ and $x \notin \langle [x, y]\rangle^G$, $exp(G) = p$, we have $|\langle [x, y]\rangle^G| = p^m$. Then $\langle x\rangle^G = \langle [x, y]\rangle^G \rtimes \langle x\rangle$. Similarly, $\langle y\rangle^G = \langle [x, y]\rangle^G \rtimes \langle y\rangle$. Let $G_1 = \langle x\rangle^G \langle y\rangle^G$.

We claim that $G_1 \cap Z(G) = \langle [x, y]\rangle^G \cap Z(G)$ is a cyclic group of order p. By the choice of $[x, y]$, we see $|\langle [x, y]\rangle^G \cap Z(G)| = p$. It is easy to see that $|\langle x\rangle^G \cap Z(G)| = |\langle y\rangle^G \cap Z(G)| = p$. Suppose that $|G_1 \cap Z(G)| \geqslant p^2$, then there exist $1 \leqslant i, j \leqslant p-1$ such that $g = cx^i y^j \in G_1 \cap Z(G)$, where $c \in \langle [x, y]\rangle^G$. Hence $[cx^i y^j, y] = 1$. Let $\bar{G} = G/Z_{m-1}(G)$, and $\bar{c} = cZ_{m-1}(G)$, $\bar{x} = xZ_{m-1}(G)$, $\bar{y} = yZ_{m-1}(G)$. Then $[\bar{c}\bar{x}^i \bar{y}^j, \bar{y}] = 1$, and then $[\bar{x}, \bar{y}]^i = 1$. Since $(i, p) = 1$, we see $[x, y] \in Z_{m-1}(G)$, a contradiction.

Suppose there exists $w \in Z_2(G) - Z(G)$ such that $\langle w\rangle^G \cap G_1 = 1$. Then $\langle G_1, \langle w\rangle^G\rangle = G_1 \times \langle w\rangle^G$. Let $g = xw$. Since $[g, y] = [x, y]^w [w, y] = [x, y]$, we see that $\langle [x, y]\rangle^G \leqslant \langle g\rangle^G$. As $exp(G) = p$, $\langle [x, y]\rangle^G \cap \langle g\rangle = 1$. By the choice of w, there exists $b \in G$ such that $[w, b] \neq 1$. Let $w_1 = [w, b]$. Then $[g, b] = [x, b]^w [w, b] = [x, b]^w w_1$. Note that $[x, b]^w \in G_1$. If $[g, b] \in \langle [x, y]\rangle^G \rtimes \langle g\rangle$, then $[x, b]^w w_1 = sg^k = sx^k w^k$, where $s \in \langle [x, y]\rangle^G$, k is an integer with $(k, p) = 1$. Hence $w^k w_1^{-1} = (sx^k)^{-1}[x, b]^w$. Since $w^k w_1^{-1} \in \langle w\rangle^G$ and $(sx^k)^{-1}[x, b]^w \in G_1$, we have $w^k w_1^{-1} = 1$, and $w^k = w_1$. Thus $w \in \langle [w, b]\rangle^G$, a contradiction. Hence $[g, b] \notin \langle [x, y]\rangle^G \rtimes \langle g\rangle$. But $[g, b] \in \langle g\rangle^G$. So we see that $|\langle g\rangle^G : \langle g\rangle| > p^m$, which contradicts to $G \in BI(p^m)$. Therefore, $\langle w\rangle^G \cap G_1 \neq 1$ for any $w \in Z_2(G) - Z(G)$. Hence $\langle a\rangle^G \cap G_1 \neq 1$ for any $a \in G - Z(G)$. Let $A = G_1 \cap Z(G)$. Then $A \leqslant \langle a\rangle^G$. So $G/A \in BI(p^{m-1})$. By Theorem 4.1, $cl(G/A) \leqslant m-1$. Obviously $[xA, yA] \in Z_{m-1}(G/A) - Z_{m-2}(G/A)$ and then $cl(G/A) = m-1$. By induction, $|(G/A)'| = p^{m-1}$, and then $|G'| = p^m$. $\square$

Now we give an interesting example of $BI(p^m)$-group.

Example 4.4. Let $H_1 = \langle x\rangle \times \langle a_1\rangle \times \cdots \times \langle a_m\rangle$, where $|x| = p^2$, $|a_i| = p$ for $i = 1, \ldots, m$. Let $b_1, \ldots, b_{m+1} \in Aut(H_1)$ defined by $x^{b_i} = xa_i$, $a_j^{b_i} = a_j$ for $i = 1, \ldots, m$, $j = 1, \ldots, m$, $x^{b_{m+1}} = x^{1+p}$, $a_j^{b_{m+1}} = a_j$ for $j = 1, \ldots, m$. Let $H_2 = \langle b_1, \ldots, b_{m+1}\rangle$, and $H = H_1 \rtimes H_2$. Obviously, $H_2 = \langle b_1\rangle \times \cdots \times \langle b_{m+1}\rangle$ is an elementary abelian p-group of order p^{m+1}, and $H_1 = \langle x\rangle^H$. Then $H \in BI(p^m)$.

Proposition 4.5. *Let $T = G \times H$, where $G \in BI(p^m)$, H is the group in Example 4.4. If $T \in BI(p^m)$, then G is an elementary abelian p-group.*

Proof. All symbols in Example 4.4 have the same meaning in the following proof.

We shall first prove that $exp(G) = p$. Suppose $g \in G$ with $|g| = p^2$. It is clear that $[gx, h] = [x, h]$ for any $h \in H$. Let $A = \langle x^p\rangle \times \langle a_1\rangle \times \cdots \times \langle a_m\rangle$. By the definition of H, we see $A \leqslant \langle gx\rangle^T$. Then $|\langle gx\rangle^T : \langle gx\rangle| > p^m$, which contradicts to $T \in BI(p^m)$.

Now we prove G is abelian. Otherwise, there exists $g_1 \in Z_2(G) - Z(G)$. Then $\langle g_1\rangle^G = \langle g_1\rangle \times [\langle g_1\rangle, G]$. $[\langle g_1\rangle, G] \neq 1$ by the choice of g_1. Let $B = \langle a_1\rangle \times \cdots \times \langle a_m\rangle$. As above, $[\langle g_1\rangle, G] \leqslant \langle g_1\rangle^T$, $B \leqslant \langle g_1 x\rangle^T$. Obviously, $[\langle g_1\rangle, G] \cap \langle g_1 x\rangle = 1$, $B \cap \langle g_1 x\rangle = 1$. Let $A_1 = [\langle g_1\rangle, G] \times \langle g_1 x\rangle$, $A_2 = B \times \langle g_1 x\rangle$. Then $A_1 \cap A_2 = \langle g_1 x\rangle$, and $|A_1 A_2| > p^{m+2}$. Since $A_i \leqslant \langle g_1 x\rangle^T$, we have $|\langle g_1 x\rangle^T : \langle g_1 x\rangle| > p^m$, a contradiction. $\square$

Proposition 4.6. *Let $T = G \times H$, where G is an elementary p-group and $H \in BI(p^m)$. Then $T \in BI(p^m)$.*

Proof. Let $h \in H, g \in G$. Then $\langle gh\rangle^T = \langle gh, [\langle gh\rangle, T]\rangle = \langle gh, [\langle gh\rangle, H]\rangle = \langle gh, [\langle h\rangle, H]\rangle$. It is clear that $\langle h\rangle^H = \langle h, [\langle h\rangle, H]\rangle$, $[\langle h\rangle^H, H] \leqslant \langle h\rangle^H$, $[\langle h\rangle, H] \leqslant [\langle h\rangle^H, H]$, and $[\langle h\rangle^H, H] \lhd H$. Hence $\langle h\rangle^H = \langle h\rangle[\langle h\rangle^H, H]$ and $\langle gh\rangle^T \leqslant \langle gh\rangle[\langle h\rangle^H, H]$. Since $h \notin [\langle h\rangle^H, H]$ for H is a p-group, it follows that $\langle h\rangle \cap [\langle h\rangle^H, H] = \langle h^p\rangle \cap [\langle h\rangle^H, H]$. Note that $(gh)^p = h^p$ for G is elementary. We see $\langle h\rangle \cap [\langle H\rangle^H, H] \leqslant$

$\langle gh\rangle \cap [\langle h\rangle^H, H]$. Obviously $\langle gh\rangle \cap [\langle h\rangle^H, H] \leqslant \langle h\rangle \cap [\langle h\rangle^H, H]$. Thus $\langle h\rangle \cap [\langle h\rangle^H, H] = \langle gh\rangle \cap [\langle h\rangle^H, H]$, and then $|\langle h\rangle[\langle h\rangle^H, H]| = |\langle gh\rangle[\langle h\rangle^H, H]|$, which implies $|\langle h\rangle^H| \geqslant |\langle gh\rangle^T|$. So we have that $|\langle gh\rangle^T : \langle gh\rangle| \leqslant |\langle h\rangle^H : \langle h\rangle| \leqslant p^m$. $\square$

Combining Propositions 4.5, 4.6, we get the following theorem.

Theorem 4.7. *Let $G \in BI(p^m)$. Then $G \times H \in BI(p^m)$ for every $H \in BI(p^m)$ if and only if G is an elementary abelian p-group.*

By Theorem 4.7, the direct product of two $BI(p^m)$-groups may not be a $BI(p^m)$-group. The following example tell us the direct product of two $BI(p^m)$-groups may be a $BI(p^{3m})$-group.

Example 4.8. Let $A = \langle a_1\rangle \times \langle a_2\rangle$, where $|a_1| = p^{2m}$, $|a_2| = p^m$. Let $b_1, b_2 \in Aut(A)$ defined by $a_1^{b_1} = a_1a_2$, $a_2^{b_1} = a_2$, $a_1^{b_2} = a_1^{p^m+1}$, $a_2^{b_2} = a_2$. $B = \langle b_1, b_2\rangle$. Then $|b_1| = |b_2| = p^m$, and $B = \langle b_1\rangle \times \langle b_2\rangle$. Let $G = A \rtimes B$, then $G_1 = \langle a_1, a_2, b_1, b_2\rangle \in BI(p^m)$ and $\langle a_1\rangle^{G_1} : \langle a_1\rangle| = p^m$. Let G_2 be an isomorphic image of G_1, x_1, x_2, y_1, y_2, be the image of a_1, a_2, b_1, b_2, respectively. Hence $G_2 = \langle x_1, x_2, y_1, y_2\rangle \cong G_1 \in BI(p^m)$. Let $G = G_1 \times G_2$. Then $|\langle a_1x_1\rangle^G : \langle a_1x_1\rangle| = p^{3m}$, and $G \in BI(p^{3m})$.

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